Introduction to Robotics

Potential Functions, aka *May the Force be with you*

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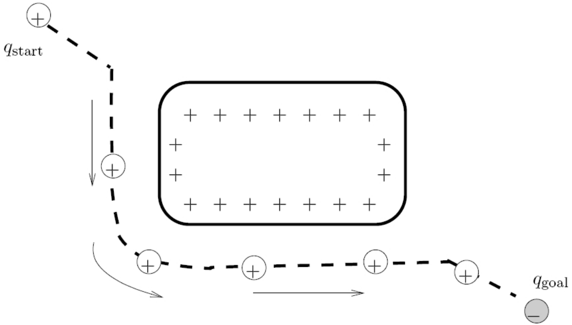
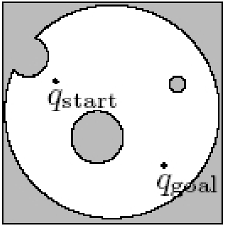
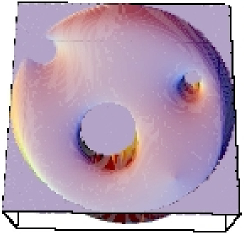
Basic Idea

- Suppose the goal is a point $g \in \mathbb{R}^2$
- Suppose the robot is a point $r \in \mathbb{R}^2$
- Think of a spring drawing the robot toward the goal and away from obstacles
- Can also think of like and opposite charges



- Think of the goal as the bottom of a bowl
- The robot is at the rim of the bowl
- What will happen?





- Both the spring and bowl analogies are ways of storing *potential energy*
- The robot moves to a lower-energy configuration

A potential function is a function $U : \mathbb{R}^n \rightarrow \mathbb{R}$

Energy is minimized by following the *negated gradient* of the potential energy function

$$\textit{gradient at } q \in \mathbb{R}^n: \quad \nabla U(q) = \left[\frac{\partial U}{\partial q_1}(q), \ldots, \frac{\partial U}{\partial q_n}(q)\right]^T$$

We can now think of a *vector field* over the space of all q's

- the robot looks at the vector at its current position and goes in that direction

Attractive + Repulsive Potentials

Desired objectives

- robot moves toward the goal (attractive potential)
- robot stays away from the obstacles (repulsive potential)

$$U(q) = U_{att}(q) + U_{rep}(q)$$

Attractive Potential: Conic Potential

Attractive potential: $U_{att}(q)$

- monotonically increasing with distance from q_{goal}
- example: conic potential (scaled distance to goal, $\zeta > 0$ scaling factor)

$$U_{att}(q) = \zeta \; ||q, q_{goal}||$$

- what's the gradient?

$$\nabla U_{att}(q) = \frac{\zeta}{||q, q_{goal}||}(q - q_{goal})$$

- what's the magnitude of the gradient at q?

$$||\nabla U_{att}(q)|| = \begin{cases} \zeta, & q \neq q_{goal} \\ undefined, & q = q_{goal} \end{cases}$$

- conic potential has discontinuity at q_{goal}

Attractive potential: $U_{att}(q)$

- monotonically increasing with distance from q_{goal}
- preference:
  continuously differentiable + magnitude decreases as robot approaches q_{goal}
- example: quadratic potential ($\zeta > 0$ scaling factor)

$$U_{att}(q) = \frac{1}{2}\zeta \;||q, q_{goal}||^2$$

- what's the gradient?

$$\nabla U_{att}(q) = \zeta \;(q - q_{goal})$$

- what's the magnitude of the gradient at q?

$$||\nabla U_{att}(q)|| = \zeta ||q, q_{goal}||$$

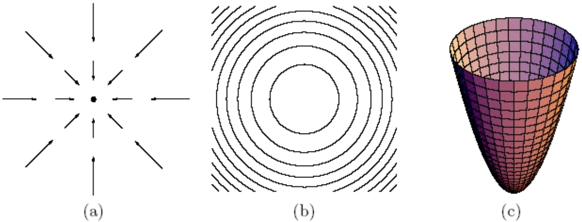


Figure: (a) Potential Field. (b) Contour Plot. (c) Quadratic Potential.

Attractive potential: $U_{att}(q)$

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- example: quadratic potential ($\zeta > 0$ scaling factor)

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- what's the gradient?

$$\nabla U_{att}(q) = \zeta \; (q - q_{goal})$$

- what's the magnitude of the gradient at q?

$$||\nabla U_{att}(q)|| = \zeta ||q, q_{goal}||$$

- what happens when robot is far away from the goal?
- robot may move too fast as potential grows without bounds the further away from goal; this may produce a velocity that is too large

Attractive potential: $U_{att}(q)$

- monotonically increasing with distance from q_{goal}
- preference:
  continuously differentiable, magnitude decreases as robot approaches q_{goal}
  does not produce very large velocities
- combine conic and quadratic potentials ($\zeta > 0$ scaling factor)

$$U_{att}(q) = \begin{cases} \frac{1}{2}\zeta \, ||q, q_{goal}||^2, & \textit{if} \;\; ||q, q_{goal}|| \leq d^*_{goal} \\ d^*_{goal}\, \zeta \, ||q, q_{goal}|| - \frac{1}{2}\zeta \left(d^*_{goal}\right)^2, & \textit{if} \;\; ||q, q_{goal}|| > d^*_{goal} \end{cases}$$

(d^*_{goal}: threshold from goal where planner switches between conic and quadratic potentials)

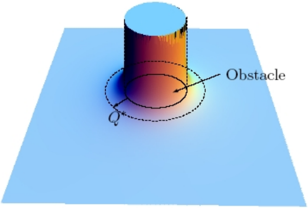
- what's the gradient? is it well defined at the boundary?

$$\nabla U_{att}(q) = \begin{cases} \zeta \, (q - q_{goal}), & \textit{if} \;\; ||q, q_{goal}|| \leq d^*_{goal} \\ d^*_{goal}\, \zeta \, (q - q_{goal}) / ||q, q_{goal}||, & \textit{if} \;\; ||q, q_{goal}|| > d^*_{goal} \end{cases}$$

Repulsive Potential

Repulsive potential: $U_{rep}(q)$

- the closer the robot is to an obstacle, the stronger the repulsive force should be
- robot keeps track of closest obstacle
- there is a threshold so robot can ignore far away obstacles



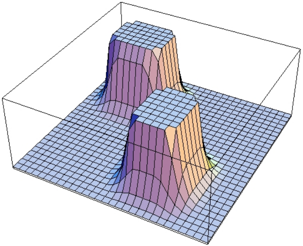
Repulsive potential: $U_{rep}(q)$

- the closer the robot is to an obstacle, the stronger the repulsive force should be

$$U_{rep}(q) = \begin{cases} \frac{1}{2}\eta \left(\frac{1}{D(q)} - \frac{1}{d^*_{obst}} \right)^2, & \text{if } D(q) \leq d^*_{obst} \\ 0, & \text{otherwise} \end{cases}$$

$$\nabla U_{rep}(q) = \begin{cases} \eta \left(\frac{1}{d^*_{obst}} - \frac{1}{D(q)} \right) \frac{1}{(D(q))^2} \nabla D(q), & \text{if } D(q) \leq d^*_{obst} \\ 0, & \text{otherwise} \end{cases}$$

- $D(q)$: distance to the closest obstacle; $\eta > 0$ scaling factor
- d^*_{obst}: threshold to allow the robot to ignore obstacles far away from it



Repulsive potential: $U_{rep}(q)$

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- $D(q)$: distance to the closest obstacle; $\eta > 0$ scaling factor
- d^*_{obst}: threshold to allow the robot to ignore obstacles far away from it
- what happens around points that are two-way equidistant from obstacles?

  D is nonsmooth $\Longrightarrow$ path may oscillate

Repulsive potential: $U_{rep}(q)$

- minimum distance to i-th obstacle

$$d_i(q) = \min_{c \in \texttt{Obstacle}_i} d(q, c)$$

- for convex obstacles (c is closest point to q)

$$\nabla d_i(q) = \frac{q - c}{||q, c||}$$

- repulsive potential for each obstacle

$$U_{rep_i}(q) = \begin{cases} \frac{1}{2}\eta \left(\frac{1}{d_i(q)} - \frac{1}{d^*_{obst_i}}\right)^2, & if \ d_i(q) \leq d^*_{obst_i} \\ 0, & otherwise \end{cases}$$

- overall repulsive potential as sum of obstacle potentials

$$U_{rep}(q) = \sum_i U_{rep_i}(q)$$

repeat until gradient is zero (or its magnitude very small)

- take small step in the direction opposite the gradient

Pseudocode

$q \leftarrow q_{init}$
while $||\nabla U(q)|| > \epsilon$ **do**
    $q \leftarrow q - \alpha \nabla U(q)$

- $\epsilon > 0$: small constant to ensure termination criteria
- $\alpha > 0$: step size (doesn't have to be constant)

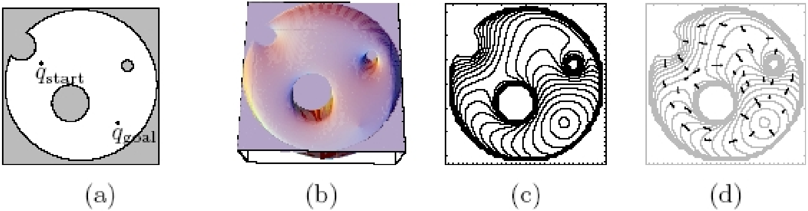


Figure: (a): Configuration space with gray obstacles. (b) Potential function energy surface. (c) Contour plot for energy surface. (d) Gradient vectors for potential function.

Gradient Descent: Moving Opposite to the Gradient

repeat until gradient is zero (or its magnitude very small)

- take small step in the direction opposite the gradient

Pseudocode

1: $q \leftarrow q_{init}$
2: **while** $||\nabla U(q)|| > \epsilon$ **do**
3: $\quad q \leftarrow q - \alpha \nabla U(q)$

- $\epsilon > 0$: small constant to ensure termination criteria
- $\alpha > 0$: step size (doesn't have to be constant)

Weaknesses of Gradient Descent

- it is relatively slow close to the minimum
- it might 'zigzag' down valleys

Better Methods

- Broyden-Fletcher-Goldfarb-Shanno (BFGS) method

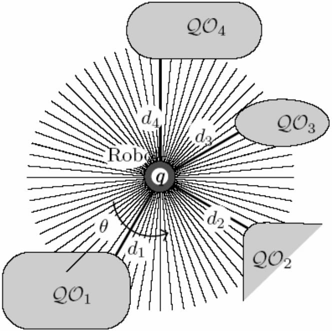
  . . . but more complex to implement

- Robot knows goal position
- Robot does not know where obstacles are located
- Robot has range sensor and can determine its own position

$U_{att}(q)$ can be easily computed since goal position is known

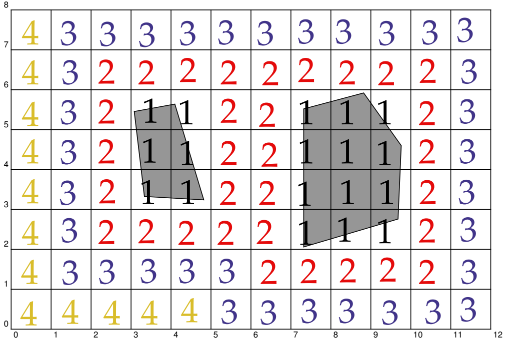
$U_{rep}(q)$ approximate it via data from range sensor

- $D(q)$: approximated as the global minimum of the raw distance function ρ
- $d_i(q)$: approximated as local minima with respect to θ in $\rho(q, \theta)$



$U_{rep}(q)$:

- discretize space by imposing a grid (define cell neighbors 4- or 8-connectivity)
- label with 1 cells that are partially or fully occupied by obstacles
- label with 2 all unlabeled cells neighboring 1-labeled cells
- ...
- label with n all unlabeled cells neighboring $(n-1)$-labeled cells
- stop when all cells have been labeled

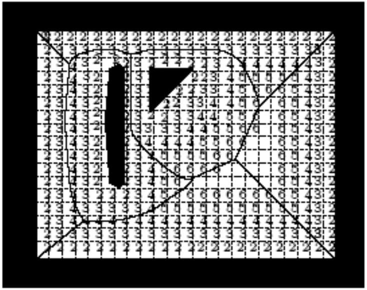


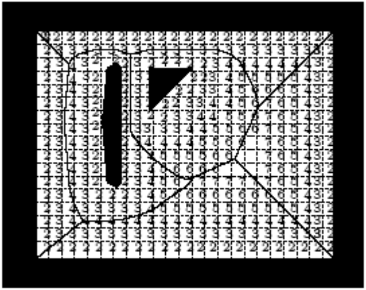
- gradient from each cell points to a neighbor with lowest label

Brushfire Algorithm – Compute Distances on a Grid

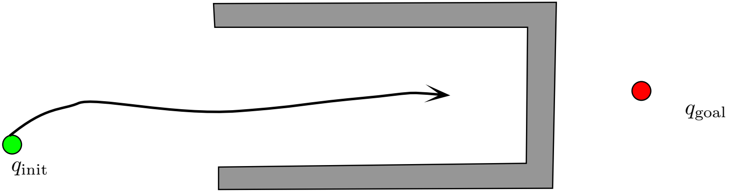
$U_{rep}(q)$:

- discretize space by imposing a grid (define cell neighbors 4- or 8-connectivity)
- label with 1 cells that are partially or fully occupied by obstacles
- label with 2 all unlabeled cells neighboring 1-labeled cells
- ...
- label with n all unlabeled cells neighboring $(n-1)$-labeled cells
- stop when all cells have been labeled
- can planner get stuck?





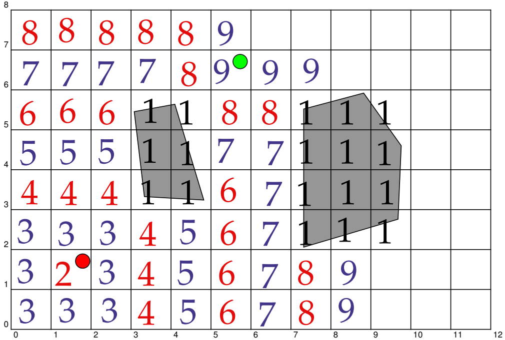
Gradient descent algorithms may get stuck in local minima



Two approaches to avoid local-minima problem

- do something different than gradient descent to overcome/avoid local minima
- define potential function so that there is only one global minimum

- similar to Brushfire algorithm discretize space by imposing a grid
- label with 1 cells that are partially or fully occupied by obstacles
- label with 2 cell where goal is located
- label with 3 all unlabeled cells neighboring 2-labeled cells
- ...
- label with n all unlabeled cells neighboring $(n-1)$-labeled cells
- stop when init cell (green circle) has been labeled



- each time move to neighboring non-obstacle cell with lowest label

How can we deal with rigid bodies and manipulators?

- Think of gradient vectors as forces
- Define forces in workspace W (which is $\mathbb{R}^2$ or $\mathbb{R}^3$)
- "Lift up" forces in configuration space Q

Relationship between Forces in the Workspace and Configuration Space

- point $x \in W$ in workspace related to configuration $q \in Q$ via forward kinematics

$$x = \mathrm{FK}(q)$$

- "virtual work" principle: work (or power) is a coordinate-independent quantity
- in workspace, power done by a force f is $f^T \dot{x}$
- in configuration space, power done by a force u is $u^T \dot{q}$
- mapping from workspace forces to configuration space forces done via Jacobian $J = \partial \mathrm{FK} / \partial q$ of the forward kinematic function

$$\begin{aligned} f^T \dot{x} &= u^T \dot{q} \quad \textit{(by the "virtual work" principle)} \\ \Rightarrow f^t J \dot{q} &= u^T \dot{q} \quad \textit{(by Jacobian property } \dot{x} = J\dot{q}\textit{)} \\ \Rightarrow f^T J &= u^T \\ \Rightarrow J^T f &= u \end{aligned}$$

Potential Functions for Rigid-Body Robots

- Pick control points $r_1, \ldots, r_n$ on the robot in its initial placement, e.g.,
  r_j could be selected as the j-th robot vertex
- Let $\mathrm{FK}_j(q)$ denote the forward kinematics of point r_j
  example: when $q = (x, y, \theta)$ and $r_j = (x_j, y_j)$

$$\mathrm{FK}_j(q) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_j \cos\theta - y_j \sin\theta + x \\ x_j \sin\theta + y_j \cos\theta + y \end{pmatrix}$$

- Define ∇U_{att_j} in workspace for each control point r_j, and scale it appropriately, e.g.,

$$\nabla U_{att_j}(q) = \mathrm{SCALE}_{att}\left(\mathrm{FK}_j(q) - \begin{pmatrix} g_x \\ g_y \end{pmatrix}\right), \quad \text{where } (g_x, g_y) \text{ is goal center}$$

- Define $\nabla U_{rep_{i,j}}$ in workspace for each control point r_j and obstacle i, and scale it appropriately,

$$\nabla U_{rep_{i,j}}(q) = \mathrm{SCALE}_{rep}\left(\begin{pmatrix} o_{i,x} \\ o_{i,y} \end{pmatrix} - \mathrm{FK}_j(q)\right),$$

  where $(o_{i,x}, o_{i,y})$ is closest point to $\mathrm{FK}_j(q)$ on obstacle i

- Compute Jacobian

$$J_j(q) = \begin{pmatrix} \frac{\partial \mathrm{FK}_j(q)[1]}{\partial x} & \frac{\partial \mathrm{FK}_j(q)[1]}{\partial y} & \frac{\partial \mathrm{FK}_j(q)[1]}{\partial \theta} \\ \frac{\partial \mathrm{FK}_j(q)[2]}{\partial x} & \frac{\partial \mathrm{FK}_j(q)[2]}{\partial y} & \frac{\partial \mathrm{FK}_j(q)[2]}{\partial \theta} \end{pmatrix}$$

- Compute overall gradient in configuration space
  (apply Jacobian to scaled versions of the workspace gradients)

$$\nabla U_{\mathrm{cs}}(q) = \sum_j J_j^T(q) \nabla U_{att_j}(q) + \sum_j J_j^T(q) \sum_i \nabla U_{rep_{i,j}}(q)$$

Apply appropriate scaling to position and orientation components separately, i.e.,

$$move_{x,y} \leftarrow \mathrm{SCALE}_{cs}(\nabla U_{cs_{x,y}}(q)), \qquad move_\theta \leftarrow \mathrm{SCALE}_{cs}(\nabla U_{cs_\theta}(q))$$

2d chain with n revolute joints where link j has length ℓ_j

End position of the j-th link ($1 \leq j \leq n$):

$$\mathrm{FK}_j(\theta_1, \theta_2, \ldots, \theta_n) = M(\theta_1)M(\theta_2)\ldots M(\theta_j)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{where for } 1 \leq i \leq j$$

$$M(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \ell_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & \ell_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & \ell_i\sin\theta_i \\ 0 & 0 & 1 \end{pmatrix}$$

Jacobian of j-th link ($1 \leq j \leq n$):

$$J_j(\theta_1, \ldots, \theta_n) = \begin{pmatrix} \frac{\partial \mathrm{FK}_j(\theta_1,\ldots,\theta_n)[1]}{\partial\theta_1} & \ldots & \frac{\partial \mathrm{FK}_j(\theta_1,\ldots,\theta_n)[1]}{\partial\theta_j} & \overbrace{0\ldots0}^{j+1\ldots n} \\ \frac{\partial \mathrm{FK}_j(q)[2]}{\partial\theta_1} & \ldots & \frac{\partial \mathrm{FK}_j(q)[2]}{\partial\theta_j} & 0\ldots0 \end{pmatrix}, \text{where for } 1 \leq i \leq j$$

$$\frac{\partial \mathrm{FK}_j(\theta_1,\ldots,\theta_n)[1]}{\partial\theta_i} = -\sin\theta_i(ga + hb + a\ell_i) + \cos\theta_i(gb - ha + b\ell_i)$$

$$\frac{\partial \mathrm{FK}_j(\theta_1,\ldots,\theta_n)[2]}{\partial\theta_i} = -\sin\theta_i(gd + he + d\ell_i) + \cos\theta_i(ge - hd + e\ell_i)$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} = M(\theta_1)\ldots M(\theta_{i-1}), \quad \begin{pmatrix} g \\ h \\ 1 \end{pmatrix} = M(\theta_{i+1})\ldots M(\theta_j)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Potential Functions for Manipulators (cont.)

2d chain with n revolute joints where link j has length ℓ_j

- Compute $J_j(\theta_1, \ldots, \theta_n)$, $i \leq j$, using a simplified but equivalent definition

$$\frac{\partial \mathrm{FK}_j}{\partial \theta_i} = \begin{pmatrix} -\mathrm{FK}_j(\theta_1, \ldots, \theta_n)[2] + \mathrm{FK}_{i-1}(\theta_1, \ldots, \theta_n)[2] \\ \mathrm{FK}_j(\theta_1, \ldots, \theta_n)[1] - \mathrm{FK}_{i-1}(\theta_1, \ldots, \theta_n)[1] \end{pmatrix}$$

- Define ∇U_{att_n} for the end-effector and scale it appropriately:

$$\nabla U_{att_n}(\theta_1, \ldots, \theta_n) = \textsc{scale}_{att}\left(\mathrm{FK}_n(\theta_1, \ldots, \theta_n) - \begin{pmatrix} g_x \\ g_y \end{pmatrix}\right), \quad (g_x, g_y)\text{: goal center}$$

- Define $\nabla U_{rep_{i,j}}$ in workspace between the end-position of the j-th link and the i-th obstacle and scale it appropriately, e.g.,

$$\nabla U_{rep_{i,j}}(\theta_1, \ldots, \theta_n) = \textsc{scale}_{rep}\left(\begin{pmatrix} o_{i,x} \\ o_{i,y} \end{pmatrix} - \mathrm{FK}_j(\theta_1, \ldots, \theta_n)\right),$$

  $(o_{i,x}, o_{i,y})$: closest point on the i-th obstacle to the end position of the j-th link

- Compute overall gradient in configuration space

$$\nabla U_{\mathrm{cs}}(\theta_1, \ldots, \theta_n) =$$
$$\textsc{scale}\Big(\sum_j J_j^T(\theta_1, \ldots, \theta_n)\nabla U_{att_n}(\theta_1, \ldots, \theta_n) +$$
$$\sum_{i,j} J_j^T(\theta_1, \ldots, \theta_n)\nabla U_{rep_{i,j}}(\theta_1, \ldots, \theta_n)\Big)$$

Summary

Basic potential fields: attractive/repulsive forces

Path planning by following gradient of potential field

- Gradient descent (incomplete, suffers from local minima)
- Brushfire algorithm (incomplete, suffers from local minima, grid world)
- Wavefront planner (complete, grid world)

Potential Functions in Non-Euclidean Spaces

- Gradients as forces
- Lift up workspace forces to configuration space forces
- Applicable to rigid body robots and manipulators