Introduction to Robotics

Localization

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Linear Dynamical Discrete-Time System with Noise

$$x(k+1) = F(k)x(k) + G(k)u(k) + v(k)$$
$$y(k) = H(k)x(k) + w(k)$$

- $x(k) \in \mathbb{R}^n$ denotes the system state at time $t_k = t_0 + Tk$
  t_0 denotes the initial time, T denotes the time step
- $u(k) \in \mathbb{R}^m$ denotes the control input, e.g., velocity commands, torques, forces
- $y(k) \in \mathbb{R}^p$ denotes the system output, e.g., values reported by sensors

- $F(k) \in \mathbb{R}^{n \times n}$ encodes the system dynamics
- $G(k) \in \mathbb{R}^{n \times m}$ describes how the inputs drive the dynamics
- $H(k) \in \mathbb{R}^{p \times n}$ describes how states are mapped into outputs
  assumed to be full row rank for all k, although it may not be square

- $v(k) \in \mathbb{R}^n$ denotes the process noise
  assumed to be white Guassian noise with zero mean and covariance matrix $V(k)$
- $w(k) \in \mathbb{R}^p$ denotes the measurement noise
  assumed to be white Guassian noise with zero mean and covariance matrix $W(k)$

Linear Dynamical Discrete-Time System with Noise

$$x(k+1) = F(k)x(k) + G(k)u(k) + v(k)$$

$$y(k) = H(k)x(k) + w(k)$$

Objective: Determine the "best" estimate of $x(k)$ given a previous estimate $x(k-1)$ together with the known input $u(k)$ and output $y(k)$

Challenges

- Presence of the unknown and unmeasurable noise vectors $v(k)$ and $w(k)$
- State cannot in general be directly determined from the outputs because $H(k)$ may not be invertible

Approach: State estimate is constructed using the time history of the known signals $y(k)$ and $u(k)$ together with the known parameters $F(k)$, $G(k)$, $H(k)$, $V(k)$, $W(k)$

A Simple Observer

Assume that there is no noise, i.e.,

$$x(k+1) = F(k)x(k) + G(k)u(k)$$

$$y(k) = H(k)x(k)$$

Notation: $\hat{x}(k_1|0\ldots k_2)$ with $k_1 \geq k_2$ denotes the value of the state estimate at time step k_1 given the output values $y(0), \ldots, y(k_2)$

Observer follows a two-step process:

1. Prediction

$$\hat{x}(k+1|0\ldots k) = F(k)\hat{x}(k|0\ldots k) + G(k)u(k)$$

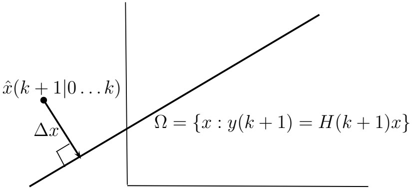
2. Update
   - Given the output $y(k+1)$, the system state is constrained to lie on the hyperplane

$$\Omega = \{x \in \mathbb{R}^n : y(k+1) = H(k+1)x\}$$

   - Choose the next estimate $\hat{x}(k+1|0\ldots k+1)$ to be the point in Ω that has the shortest distance to the prediction $\hat{x}(k+1|0\ldots k)$, i.e.,

$$\hat{x}(k+1|0\ldots k+1) = \mathrm{argmin}_{x\in\Omega}||x, \hat{x}(k+1|0\ldots k)||$$

   Why? $\hat{x}(k+1|0\ldots k)$ is close to the actual state, and the actual state must be in Ω



$$\Delta x = \hat{x}(k+1|0\ldots k+1) - \hat{x}(k+1|0\ldots k)$$

- Δx must be orthogonal to $\Omega \implies$ $a^T \Delta x = 0$ for any a that is parallel to Ω

1. A vector $a \in \mathbb{R}^n$ is parallel to Ω
$$\iff x + a \in \Omega, \forall x \in \Omega$$
$$\iff H(k+1)a = 0$$
Gives rise to the null-space, i.e.,
$$\mathrm{NullSpace}\,(H(k+1)) = \{a \in \mathbb{R}^n : H(k+1)a = 0\}$$

2. A vector $b \in \mathbb{R}^n$ is orthogonal to Ω
$$\iff a^T b = 0, \forall a \in \mathrm{NullSpace}\,(H(k+1))$$
$$\iff b \in \mathrm{RowSpace}\,(H(k+1))$$

Therefore,

$$\Delta x \text{ is orthogonal to } \Omega \iff \Delta x = H(k+1)^T \gamma, \text{for some } \gamma \in \mathbb{R}^p$$

Computing the Update (cont.)

Let ν denote the innovation error, i.e.,

$$\nu = y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k)$$

Assume for now that γ can be written as a linear function of ν, i.e.,

$$\gamma = K\nu, \quad \text{for some } K \in \mathbb{R}^{p \times p}$$

Then

$$\begin{aligned} \Delta x &= H(k+1)^T \gamma \\ &= H(k+1)^T K \nu \\ &= H(k+1)^T K \left(y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k)\right) \end{aligned}$$

Now we need to find K such that

$$y(k+1) = H(k+1)(\hat{x}(k+1|0\ldots k) + \Delta x) \implies$$

$$H(k+1)\Delta x = y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k) = \nu \implies$$

$$H(k+1)H(k+1)^T K\nu = \nu \implies$$

$$K = \left(H(k+1)H(k+1)^T\right)^{-1}$$

Does the inverse exist?

A Simple Observer: Putting it all together

Prediction

$$\hat{x}(k+1|0\ldots k) = F(k)\hat{x}(k|0\ldots k) + G(k)u(k)$$

Update

$$\begin{aligned}\hat{x}(k+1|0\ldots k+1) &= \hat{x}(k+1|0\ldots k) + \Delta x \\ &= \hat{x}(k+1|0\ldots k) + H(k+1)^T K\nu\end{aligned}$$

- $K = \left(H(k+1)H(k+1)^T\right)^{-1}$
- $\nu = (y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k))$

What are some problems with the simple observer?

- Update is always perpendicular to Ω
- Estimate errors in direction parallel to Ω are never corrected
- As a result, estimate $\hat{x}$ will not in general converge to actual state x

Probability density function:

$$Pr[a \leq X \leq b] = \int_{x=a}^{b} f(x)dx$$

Expected value for a random vector $X : S \rightarrow \mathbb{R}^n$:

$$E(X) = \int_{x \in \mathbb{R}^n} xf(x)dx$$

Variance of a scalar random variable:

$$Var(X) = E\left((X - E(X))^2\right) = E(X^2) - (E(X))^2$$

Covariance among two scalar random variables:

$$Cov(X, Y) = E\left((X - E(X))(Y - E(Y)\right) = E(XY) - E(X)E(Y)$$

Covariance matrix:

$$Cov(X) = E((X - E(X))(X - E(X))^T), \text{ i.e., } Cov_{ij}(X) = Cov(X_i, X_j)$$

Multivariate gaussian distribution with mean $\bar{X}$ and covariance matrix P:

$$f(x; \bar{X}, P) = \frac{1}{\sqrt{(2\pi)^n |P|}} e^{-\frac{1}{2}(x-\bar{X})P^{-1}(x-\bar{X})^T}$$

Assume that there is process noise but no measurement noise, i.e.,

$$x(k+1) = F(k)x(k) + G(k)u(k) + v(k)$$

$$y(k) = H(k)x(k)$$

Recall that

- $H(k)$: assumed to be full row rank for all k, although it may not be square
- $v(k)$: assumed to be white Guassian noise with zero mean and covariance matrix $V(k)$

  white here means $v(k)$ is independent of $v(k-1)$ for all k

Objective is to generate both

- a state vector estimate $\hat{x}(k|0\ldots k)$ and
- a covariance matrix estimate $P(k|0\ldots k)$

Hence

- Prediction will generate $\hat{x}(k+1|0\ldots k)$ and $P(k+1|0\ldots k)$
- Update will generate the next estimate given $\hat{x}(k+1|0\ldots k+1)$ and $P(k+1|0\ldots k+1)$

Observing with Probability Distributions: Prediction Step

Predicted state vector:

$$\hat{x}(k+1|0\ldots k) = F(k)\hat{x}(k|0\ldots k) + G(k)u(k) + E(v(k)) = F(k)\hat{x}(k|0\ldots k) + G(k)u(k)$$

Predicted covariance matrix:

$$\begin{aligned}
P(k+1|0\ldots k) &= E\left((x(k+1) - \hat{x}(k+1|0\ldots k))(x(k+1) - \hat{x}(k+1|0\ldots k)^T\right) \\
&\quad \text{substituting } x(k+1) \text{ and } \hat{x}(k+1|0\ldots k) \text{ yields} \\
&= E\left(F(k)(x(k) - \hat{x}(k|0\ldots k))(x(k) - \hat{x}(k|0\ldots k))^T F(k)^T\right. \\
&\quad \left. + 2F(k)(x(k) - \hat{x}(k|0\ldots k))v(k)^T + v(k)v(k)^T\right) \\
&= F(k)E\left((x(k) - \hat{x}(k|0\ldots k))(x(k) - \hat{x}(k|0\ldots k))^T)\right) F(k)^T + \\
&\quad E\left(v(k)v(k)^T\right) \\
&= F(k)P(k|0\ldots k)F(k)^T + V(k)
\end{aligned}$$

Chose $\hat{x}(k+1|0\ldots k+1)$ to be the most likely point in the set

$$\Omega = \{x \in \mathbb{R}^n : y(k+1) = H(k+1)x\}$$

$\implies$ Choose $x \in \Omega$ that maximizes the Gaussian distribution with mean $\hat{x}(k+1|0\ldots k)$ and covariance matrix $P(k+1|0\ldots k)$, i.e.,

$$f(x) = \frac{\exp\left(-\frac{1}{2}(x-\hat{x}(k+1|0\ldots k))P(k+1|0\ldots k)^{-1}(x-\hat{x}(k+1|0\ldots k))^T\right)}{\sqrt{(2\pi)^n|P(k+1|0\ldots k)|}}$$

$\implies$ Choose $x \in \Omega$ that minimizes

$$(x-\hat{x}(k+1|0\ldots k))P(k+1|0\ldots k)^{-1}(x-\hat{x}(k+1|0\ldots k))^T$$

Define new inner product and (Mahalanobis) distance in $\mathbb{R}^n$ as

$$\begin{aligned}\langle x_1, x_2\rangle_M &= x_1^T P(k+1|0\ldots k)^{-1} x_2 \\ ||x||_M^2 &= \langle x, x\rangle_M = x^T P(k+1|0\ldots k)^{-1} x\end{aligned}$$

Let $\Delta x = \hat{x}(k+1|0\ldots k+1) - \hat{x}(k+1|0\ldots k)$.
So we want to find $\hat{x}(k+1|0\ldots k+1)$ such that

1. $||\Delta x||_M$ is minimized
2. $(\hat{x}(k+1|0\ldots k) + \Delta x) \in \Omega$

$||\Delta x||_M$ is minimized

$\implies$ Δx is orthogonal to Ω according to inner product $\langle \cdot, \cdot \rangle_M$

$\implies$ For all $a \in \mathrm{NullSpace}(H(k+1))$

$$aP(k+1|0\ldots k)^{-1}(\Delta x) = 0$$

$\implies$ $\Delta x \in \mathrm{ColumnSpace}(P(k+1|0\ldots k)H(k+1)^T)$

$\implies$ For some $\gamma \in \mathbb{R}^p$

$$\Delta x = P(k+1|0\ldots k)H(k+1)^T\gamma$$

Let ν denote the innovation error, i.e.,

$$\nu = y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k)$$

Assume that γ can be written as a linear function of ν, i.e.,

$$\gamma = K\nu, \quad \text{for some } K \in \mathbb{R}^{p \times p}$$

Then $\Delta x = P(k+1|0\ldots k)H(k+1)^T K\nu$

$(\hat{x}(k+1|0\dots k) + \Delta x) \in \Omega$

$$
\begin{aligned}
\implies \quad & y(k+1) = H(k+1)(\hat{x}(k+1|0\dots k) + \Delta x) \\
\implies \quad & H(k+1)\Delta x = \nu \\
\implies \quad & H(k+1)P(k+1|0\dots k)H^T K\nu = \nu \\
& (\text{since also } \Delta x = P(k+1|0\dots k)H^T K\nu) \\
\implies \quad & K = (H(k+1)P(k+1|0\dots k)H(k+1)^T)^{-1}
\end{aligned}
$$

Let

$$R = P(k+1|0\dots k)H(k+1)^T K$$

Then, the update for the state vector estimate is

$$
\begin{aligned}
\hat{x}(k+1|0\dots k+1) &= \hat{x}(k+1|0\dots k) + \Delta x \\
&= \hat{x}(k+1|0\dots k) + P(k+1|0\dots k)H(k+1)^T K\nu \\
&= \hat{x}(k+1|0\dots k) + R\nu
\end{aligned}
$$

Update for the covariance matrix estimate

$$P(k+1|0\dots k+1) = P(k+1|0\dots k) - RH(k+1)P(k+1|0\dots k)$$

Prediction

$$
\begin{aligned}
\hat{x}(k+1|0\ldots k) &= F(k)\hat{x}(k|0\ldots k) + G(k)u(k) \\
P(k+1|0\ldots k) &= F(k)P(k|0\ldots k)F(k)^T + V(k)
\end{aligned}
$$

Update

$$
\begin{aligned}
\hat{x}(k+1|0\ldots k+1) &= \hat{x}(k+1|0\ldots k) + R\nu \\
P(k+1|0\ldots k+1) &= P(k+1|0\ldots k) - RH(k+1)P(k+1|0\ldots k)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu &= y(k+1) - H(k+1)\hat{x}(k+1|0\ldots k) \\
R &= P(k+1|0\ldots k)H(k+1)^T(H(k+1)P(k+1|0\ldots k)H(k+1)^T)^{-1}
\end{aligned}
$$

What are some problems with this observer?

- Since we assumed no sensor noise, the update equations can cause the covariance matrix estimate to become singular
- But if covariance matrix is singular, Gaussian distribution and Mahalanobis distance are not defined since they rely on the inverse matrix

to address these problems. . . the kalman filter

Linear Kalman Filter

$$x(k+1) = F(k)x(k) + G(k)u(k) + v(k)$$
$$y(k) = H(k)x(k) + w(k)$$

Recall that

- $v(k), w(k)$: white Guassian noise with zero mean and covariance matrix $V(k), W(k)$

Prediction (no changes from before)

$$\begin{aligned} \hat{x}(k+1|0\ldots k) &= F(k)\hat{x}(k|0\ldots k) + G(k)u(k) \\ P(k+1|0\ldots k) &= F(k)P(k|0\ldots k)F(k)^T + V(k) \end{aligned}$$

Update: Changes due to the sensor noise term $w(k)$

- Before, we knew that the constrained the next state estimate to be in Ω, so we used the equation $y(k+1) = H(k+1)\hat{x}(k+1|0\ldots k+1)$ to find $\hat{x}(k+1|0\ldots k+1)$
- Now we only know that the output is drawn from a Gaussian distribution in $\mathbb{R}^p$ with mean $y(k+1)$ and covariance matrix $W(k)$
- So will first look for the most likely output y^* given the prediction $(\hat{x}(k+1|0\ldots k), P(k+1|0\ldots k))$ together with the measured output $y(k+1)$
- After that, we can introduce the constraint $y^* = H(k+1)\hat{x}(k+1|0\ldots k+1)$ and proceed as before

- Project the prediction into output space

State space distribution with mean $\hat{x}(k+1|0\dots k)$ and covariance matrix $P(k+1|0\dots k)$ projects into a Gaussian distribution in the output space $\mathbb{R}^p$ with mean

$$\hat{y} = H(k+1)\hat{x}(k+1|0\dots k)$$

and covariance matrix

$$\begin{aligned}\hat{W} &= E\left[(\hat{y} - y(k+1))(\hat{y} - y(k+1))^T\right] \\ &= E\left[H(k+1)(\hat{x}(k+1|0\dots k) - x(k+1))(\hat{x}(k+1|0\dots k) - x(k+1))^T H(k+1)^T\right] \\ &= H(k+1)P(k+1|0\dots k)H(k+1)^T\end{aligned}$$

y^* is then the most likely point in the output space $\mathbb{R}^p$ given

- $(\hat{y}, \hat{W})$: Gaussian distribution that results from projection the state prediction
- $(y(k+1), W(k+1))$: Gaussian distribution that results from taking the measurement

y^* will be the peak of the function that results from taking their product (since distributions $(\hat{y}, \hat{W})$ and $(y(k+1), W(k+1))$ are independent)

Theorem: *The product of two Gaussians* (z_1, C_1) *and* (z_2, C_2) *is proportional to a third Gaussian* (z_3, C_3)*, where*

$$\begin{aligned} z_3 &= z_1 + C_1(C_1 + C_2)^{-1}(z_2 - z_1) \\ C_3 &= C_1 - C_1(C_1 + C_2)^{-1}C_1 \end{aligned}$$

Then

$$y^* = \hat{y} + \hat{W}(\hat{W} + W(k+1))^{-1}(y(k+1) - \hat{y})$$

We can also define

$$\Omega^* = \{x \in \mathbb{R}^n : y^* = H(k+1)x\}$$

and proceed to find $\Delta x = \hat{x}(k+1|0 \ldots k+1) - \hat{x}(k+1|0 \ldots k)$ that

- minimizes $||\Delta x||_M$ and
- satisisifes $\hat{x}(k+1|0 \ldots k+1) \in \Omega^*$

1. $||\Delta x||_M$ is minimized
$\implies$ Δx is orthogonal to Ω^* according to inner product $\langle \cdot, \cdot \rangle_M$
$\implies$ For some $\gamma \in \mathbb{R}^p$: $\Delta x = P(k+1|0\ldots k)H(k+1)^T\gamma$
Let ν be the innovation error

$$\nu = y^* - H(k+1)\hat{x}(k+1|0\ldots k)$$

Assume that γ can be written as

$$\gamma = K\nu, \quad \text{for some } K \in \mathbb{R}^{p\times p}$$

Then $\Delta x = P(k+1|0\ldots k)H(k+1)^T K\nu$

2. $(\hat{x}(k+1|0\ldots k) + \Delta x) \in \Omega$

$$\begin{aligned}
&\implies \quad y^* = H(k+1)(\hat{x}(k+1|0\ldots k) + \Delta x) \\
&\implies \quad H(k+1)\Delta x = \nu \\
&\implies \quad H(k+1)P(k+1|0\ldots k)H^T K\nu = \nu \\
&\implies \quad K = (H(k+1)P(k+1|0\ldots k)H(k+1)^T)^{-1}
\end{aligned}$$

Therefore, from (1) and (2),

$$\Delta x = P(k+1|0\ldots k)H(k+1)^T K\nu$$

where

- $K = (H(k+1)P(k+1|0\ldots k)H(k+1)^T)^{-1}$
- $\nu = y^* - H(k+1)\hat{x}(k+1|0\ldots k)$
- $y^* = \hat{y} + \hat{W}(\hat{W} + W(k+1))^{-1}(y(k+1) - \hat{y})$
- $\hat{W} = H(k+1)P(k+1|0\ldots k)H(k+1)^T$
- $\hat{y} = H(k+1)\hat{x}(k+1|0\ldots k)$

Some simplifications:

$$\begin{aligned}
K\nu &= K(y^* - H(k+1)\hat{x}(k+1|0\ldots k)) \\
&= K\hat{W}(\hat{W} + W(k+1))^{-1}(y(k+1) - \hat{y}) \\
&= (\hat{W} + W(k+1))^{-1}(y(k+1) - \hat{y})
\end{aligned}$$

Therefore, (with the shorthand notation $H \equiv H(k+1)$, $P \equiv P(k+1|0\ldots k)$)

$$\Delta x = PH^T(HPH^T + W(k+1))^{-1}(y(k+1) - H\hat{x}(k+1|0\ldots k))$$

Linear Kalman Filter: Putting it all together

Prediction

$$\begin{aligned}
\hat{x}(k+1|0\ldots k) &= F(k)\hat{x}(k|0\ldots k) + G(k)u(k) \\
P(k+1|0\ldots k) &= F(k)P(k|0\ldots k)F(k)^T + V(k)
\end{aligned}$$

Update

$$\begin{aligned}
\hat{x}(k+1|0\ldots k+1) &= \hat{x}(k+1|0\ldots k) + \Delta x \\
&= \hat{x}(k+1|0\ldots k) + \\
&\quad PH^T(HPH^T + W(k+1))^{-1}(y(k+1) - H\hat{x}(k+1|0\ldots k)) \\
P(k+1|0\ldots k+1) &= E\left[(x(k+1) - \hat{x}(k+1|0\ldots k+1))(x(k+1) - \hat{x}(k+1|0\ldots k
\end{aligned}$$

where $H \equiv H(k+1)$, $P \equiv P(k+1|0\ldots k)$

Consider a mobile robot constrained to move along a straight line.
Robot state $x = (x_r, v_r)^T$

- x_r: robot position
- v_r: robot velocity

Input control u: real-valued force applied to the robot.According to Newton's law

$$\frac{dv_r}{dt} = \frac{u}{m}$$

Approximated by the discrete time equation (T discretization rate (in seconds))

$$\frac{v_r(k+1) - v_r(k)}{T} = \frac{u(k)}{m}$$

Therefore,

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{F(k)} x(k) + \underbrace{\begin{bmatrix} 0 \\ T/m \end{bmatrix}}_{G(k)} u(k) + v(k)$$

where $v(k)$ is white Gaussian noise with zero mean and covaraince matrix V
Suppose sensor measures velocity. Then,

$$y(k+1) = \underbrace{[0, 1]}_{H(k)} x(k) + w(k)$$

where $w(k)$ is white Gaussian noise with zero mean and covaraince matrix W